

On Geometric Ergodicity of Skewed - SVCHARME models

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Abstract

Markov Chain Monte Carlo is repeatedly used to analyze the properties of intractable distributions in a convenient way. In this paper we derive conditions for geometric ergodicity of a general class of nonparametric stochastic volatility models with skewness driven by hidden Markov Chain with switching.

Keywords:

Markov switching, geometric ergodicity, irreducibility, mixture models, asymmetric stochastic volatility

1. Introduction

The asymmetry in the intertemporal relation between volatility and the stock return is the purpose of active study. In general, asymmetric effects in volatility denote, that the effects of positive returns on volatility are different from those of negative returns of a similar magnitude. Asymmetry is also sometimes referred to us as a negative relation between the ex-post volatility in the rate of returns on equity and the current value of the equity, *i.e.*, the commonly investigated leverage effect. It is therefore necessary to stress,

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that although leverage effect exhibits asymmetry, not all asymmetric effects display leverage.

Two different approaches are of considerable interest, namely the conditional and stochastic volatility models. In the class of conditional volatility, ARCH specifications that have been developed to capture asymmetric effects are: the Exponential GARCH (EGARCH) model of Nelson (1991) [18] and the GJR model of Glosten, Jagannathan and Runkle (1993) [8]. In the manner depicted earlier, EGARCH model can describe leverage whereas the GJR model can capture asymmetric effects but not leverage. On the other hand, stochastic volatility (SV) models have become a natural alternative to time-varying volatility of the ARCH family. The asymmetric property within the stochastic volatility framework is established on a straightforward correlation between the innovations in both returns and volatility. Unlike ARCH type models, the SV models allow volatility dynamics to follow some latent stochastic processes based on volatility's unobservable nature. Basic stochastic volatility process (BSV) was originally developed by Hull and White (1987) [13] and Jacquier et al. (2004)[14] in continuous time framework as a mode to capture negative correlation between the innovation terms in the Black-Scholes option pricing formula. This basic version of the SV model assumes stationary error disturbances and uncorrelated Gaussian white noise processes (Harvey and Ruiz, 1993 [10]). In empirical research, extensions of a simple discrete time model due to Taylor (1986)[23] have been analyzed by Wiggins (1987)[26] and Harvey and Shephard (1996)[20] in order to accommodate the direct correlation. This generalization, based on the immediate correlation between the innovation is known as the SV with leverage (SV-L) model (Asai and McAleer, 2006 [1]). Other asymmetric models were suggested by Danielsson (1994)[5], which was similar in the spirit to that of the EGARCH model. Nelson (1991)[18] used the absolute value function to capture the sign and magnitude of the previous value of normalized returns in accommodating asymmetric behaviour into an ARCH-type model. Danielsson (1994)[5] used the absolute value function as in Nelson (1991) [18], but incorporated the observed return into the SV specification as it is not computationally straightforward in the SV framework to incorporate the normalized disturbances, in a SV with leverage and size effects (SV-LSE) model. So, Li and Lam (2002) [21] considered a different type of threshold effects model in which the breaks in the constant and autoregressive parameter in the SV equation depend on the signs of the previous returns. An alternative form of asymmetry can be based on threshold effects, as pro-

posed in Glosten, Jagannathan and Runkle (1992)[8] in the context of conditional volatility models. A variety of symmetric and asymmetric, univariate and multivariate, conditional and stochastic volatility models is analyzed in McAleer (2005)[16]. Excess kurtosis and skewness in the residuals of the basic SV model has triggered the use of either the mixtures of normals or the central t-distribution (Geweke, 1994 [7]; Shephard and Pitt, 1997 [19]). Hansen (1994)[9] has considered skewness in a GARCH model using skew t-distribution errors allowing for both skewness and heavy tails to co-exist in a time-varying volatility specification. Also, Cappuccio et al. (2004)[4] have managed to generate skewness and excess kurtosis in the conditional distribution of returns by assuming a skew-GED distribution. Volatility asymmetry, initially interpreted as the leverage effect, is expressed by the correlation of the returns with the future volatility. Under this assumption, when mean returns are negatively correlated with contemporaneous volatility, the negative (positive) return shocks are associated with increased (decreased) future volatility. Harvey and Shephard (1996)[20] have developed the asymmetric SV (ASV-t) model, which guarantees the existence of the martingale property. Recently, Wang et al. (2011)[25] have developed a heavy-tailed SV model with leverage effect, where a bivariate t-distribution, in the form of a scale mixture of normal, is used to model the error innovations of the return and volatility equations. Tsiotas (2012)[24] introduced generalised ASV models that incorporate financial data stylized facts including excess kurtosis, skewness, and volatility asymmetry, using three alternative distributions: the noncentral-t (NCT), the skew-normal (SN), and the skew-t (ST) distributions. These ASV models incorporate both the Gaussian and the non-Gaussian assumption in the same specification, while taking into account the leverage effect. The asymmetric central-t distributed SV (hereafter ASV-t) model nests in the asymmetric noncentral-t distributed SV (hereafter ASV-nct) model. The other one is the asymmetric skewnormal SV (hereafter ASV-sn) model. Finally, in the asymmetric skew-t SV (hereafter ASV-st) model both excess kurtosis and skewness co-exist. The first multivariate SV model was proposed by Harvey, Ruiz and Shephard (1994)[11], who specified the model in terms of instantaneous correlations in the mean and volatility equations. However, their estimation technique was based on the inefficient quasi-maximum likelihood (QML) procedure. Danielsson (1998)[6] suggested a multivariate SV-L model based on the specification considered by Harvey, Ruiz and Shephard (1994)[11], but only estimated a symmetric version of the model. Shephard (1996) [20] proposed a one factor multivariate SV model.

Parametric and log-volatility estimation of the asymmetric stochastic volatility models is usually implemented in a Markov Chain Monte Carlo (MCMC) set-up. By overcoming the problem of likelihood non-existence in a close form solution, because of normality departures in the observed process, the Metropolis-Hastings algorithm is common in the MCMC engine. These algorithms provide a reliable structure to explore the intractable distributions. In any MCMC analysis, the convergence rate of the associated Markov Chain is of practical and theoretical importance. Geometric ergodicity is essential in convergence of Markov Chains to their stationary distributions. A geometrically ergodic chain converges to its target distribution at a geometric rate. In addition to ensuring the rapid convergence required for useful simulation, geometric ergodicity is a key sufficient condition for the existence of Central Limit Theorems and consistent estimators of Monte Carlo standard errors, *i.e.*, geometric ergodicity justifies the applicability of the Central Limit Theorems to ergodic averages along the path of the chain. The main aim of this paper is to provide a formal proof of geometric ergodicity for general non-parametric model, that incorporates both the asymmetry and is the natural extension to CHARME models provided by Stockis et al. 2010.

2. The Skewed Stochastic Volatility-CHARME model

Consider first a nonlinear p th order autoregressive process defined by

$$X_t = m(X_{t-1}, \dots, X_{t-p}) + \sigma(X_{t-1}, \dots, X_{t-p})\epsilon_t + A(X_{t-1}, \dots, X_{t-p})\iota_t^2, \quad (1)$$

where m , σ and A are unknown, real-valued Borel measurable functions. The ϵ_t and ι_t are mutually independent identically distributed zero-mean innovations. Since, it is unremarkably not realistic to postulate that the observed process has the same trend function m , volatility function σ or skewness function A at each time instant, we consider a class of nonparametric time-series models, where between random change points, the process we observe is piecewise stationary. The dynamics of $\{X_t\}$ is driven by a hidden Markov chain $\{Q_t\}$ with values in a finite set $\{1, 2, \dots, K\}$. Our model is defined as follows:

$$X_t = \sum_{k=1}^K S_{tk} \left(m_k(X_{t-1}, \dots, X_{t-p}) + \sigma_k(X_{t-1}, \dots, X_{t-p})\epsilon_t + A_k(X_{t-1}, \dots, X_{t-p})\iota_t^2 \right), \quad (2)$$

where

$$S_{tk} = \begin{cases} 1, & \text{for } Q_t = k \\ 0, & \text{otherwise} \end{cases}$$

m_k , σ_k and A_k , $k = 1, 2, \dots, K$ are unknown functions, ϵ_t and ι_t are independent and identically distributed random variables with mean zero and they are mutually independent.

2.1. Model definition

We make the following assumptions, adopting Stockis et al. [22] point of view:

Assumption 2.1. *The process $\{Q_t\}$ is a first - order strictly stationary Markov chain which is irreducible and aperiodic with probability distribution $(\pi_1, \pi_2, \dots, \pi_K)$ and transition probability matrix $A = \{a_{ij}\}_{1 \leq i, j \leq K}$.*

Assumption 2.2. *Let $\mathbb{G}_{t-1} = \sigma\{X_s, s \leq t-1\}$ be the σ -algebra generated by $\{X_s, s \leq t-1\}$ and let $G_{t-1} \in \mathbb{G}_{t-1}$. Then $P(Q_t = j | Q_{t-1} = i, G_{t-1}) = P(Q_t = j | Q_{t-1} = i)$ for all i, j . In other words, the hidden process Q_t is independent of the past observations of $\{X_t\}$ given its own past.*

Assumption 2.3. *Q_t is uncorrelated with the ϵ_t and ι_t , given $(Q_{t-1}, X_{t-1}, X_{t-2}, \dots)$.*

Assumption 2.4. *Both ϵ_t and ι_t are independent of X_{t-1}, X_{t-2}, \dots .*

Assumption 2.5. *The functions m_k , σ_k and A_k are bounded on compact sets for all k .*

Assumption 2.6. *The iid random variables ϵ_t have a continuous and everywhere positive density f and the iid random variables ι_t have a continuous and everywhere positive density g .*

Without the loss of generality, we restrict ourselves to the case $p = 1$, i.e., m_k , σ_k and A_k are functions on the real line. We assume that $A_k(x) > 0$. We should make the following assumptions

Assumption 2.7. *The iid random variables ϵ_t and ι_t have mean zero and variance equal to 1. Moreover, there exists the fourth moment $E(\iota_t^4) = \kappa$.*

Assumption 2.8.

$$\max_{i \in \{1, 2, \dots, K\}} \lim_{|x| \rightarrow +\infty} \sup \frac{\sum_{k=1}^K a_{ik} (m_k^2(x) + \sigma_k^2(x) + \kappa A_k^2(x) + 2m_k(x)A_k(x))}{x^2} < 1$$

Let $S_t = (S_{t1}, S_{t2}, \dots, S_{tK})^T$. We conclude that, under assumptions 1 – 4, the process

$$Z_t = (S_t, X_t)^T,$$

which represents the transformed mixture process, is a Markov chain as well.

Theorem 2.1. *Under Assumptions 1 – 8, the process $\{Z_t\}$ is geometrically ergodic.*

Proof. We shall prove that the conditions of Theorem 15.0.1 (iii) of Meyn and Tweedie [17] are satisfied.

1. The process Z_t is φ -irreducible if we take φ as a product of the stationary probability distribution measure of $\{Q_t\}$ on $\{1, 2, \dots, K\}$ and the Lebesgue measure on \mathbb{R} . Let $A = A_1 \times A_2$ be such that $\varphi(A) > 0$. A_1 contains at least one integer k between 1 and K . It suffices to prove that there exists t such that for all k and l

$$P((S_{t+1}, X_{t+1}) \in \{e_k\} \times A_2 | S_1 = e_l, X_1 = x) > 0$$

with e_k denoting a unit vector with the k th component equal to 1.

We obtain

$$\begin{aligned} P((S_2, X_2) \in \{e_k\} \times A_2 | S_1 = e_l, X_1 = x) &= \\ P(Q_2 = k, X_2 \in A_2 | S_1 = e_l, X_1 = x) &= \\ P(X_2 \in A_2 | Q_2 = k, S_1 = e_l, X_1 = x) P(Q_2 = k | Q_1 = l, X_1 = x) &= \\ a_{lk} P(m_k(x) + \sigma_k(x)\epsilon_2 + A_k(x)\iota_2^2 \in A_2) &= \end{aligned}$$

$$\begin{aligned} a_{lk} \int_{A_2} \int_{\mathbb{R}} \frac{1}{\sigma_k(x)} f\left(\frac{u - y - m_k(x)}{\sigma_k(x)}\right) \frac{g\left(\sqrt{\frac{y}{A_k(x)}}\right) + g\left(-\sqrt{\frac{y}{A_k(x)}}\right)}{2\left(\sqrt{\frac{y}{A_k(x)}}\right)} \mathbb{I}_{\{y>0\}} dy du \\ = a_{lk} h_k(x), \end{aligned}$$

where $h_k(x) > 0$.

The one but last equality we obtain, because the density of $m_k(x) + \sigma_k(x)\epsilon_2$ is

$$\bar{f}(y) = \frac{1}{\sigma_k(x)} f\left(\frac{y - m_k(x)}{\sigma_k(x)}\right)$$

and the density of $A_k(x)\iota_2^2$ is

$$\bar{g}(y) = \frac{g\left(\sqrt{\frac{y}{A_k(x)}}\right) + g\left(-\sqrt{\frac{y}{A_k(x)}}\right)}{2\left(\sqrt{\frac{y}{A_k(x)}}\right)} \mathbb{I}_{\{y>0\}}.$$

Then we use a well-known formula for convolution of the distributions, *i.e.*,

$$(\bar{f} * \bar{g})(y) = \int_{\mathbb{R}} \bar{f}(u - y) \bar{g}(y) du.$$

Similarly,

$$\begin{aligned} P((S_3, X_3) \in \{e_k\} \times A_2 | S_1 = e_l, X_1 = x) &= \\ P(Q_3 = k, X_3 \in A_2 | S_1 = e_l, X_1 = x) &= \\ P(X_3 \in A_2 | Q_3 = k, Q_1 = l, X_1 = x) P(Q_3 = k | Q_1 = l, X_1 = x) &= \\ \sum_{j=1}^K a_{lj} a_{jk} \int_{A_2} \int_{\mathbb{R}} P(x, dy | Q_3 = k, Q_1 = j) P(y, du | Q_3 = k, Q_1 = j) &= \\ \sum_{j=1}^K a_{lj} a_{jk} h_{jk}(x) \end{aligned}$$

where

$$\begin{aligned} h_{jk}(x) &= \int_{A_2} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\sigma_k(y)} f\left(\frac{u - v - m_k(y)}{\sigma_k(y)}\right) \frac{g\left(\sqrt{\frac{v}{A_k(y)}}\right) + g\left(-\sqrt{\frac{v}{A_k(y)}}\right)}{2\left(\sqrt{\frac{v}{A_k(y)}}\right)} \mathbb{I}_{\{v>0\}} \cdot \\ &\cdot \frac{1}{\sigma_j(x)} f\left(\frac{y - w - m_j(x)}{\sigma_j(x)}\right) \frac{g\left(\sqrt{\frac{w}{A_j(x)}}\right) + g\left(-\sqrt{\frac{w}{A_j(x)}}\right)}{2\left(\sqrt{\frac{w}{A_j(x)}}\right)} \mathbb{I}_{\{w>0\}} dw dv dy du \end{aligned}$$

and $h_{jk}(x) > 0$.
Finally, we obtain

$$P((S_{t+1}, X_{t+1}) \in \{e_k\} \times A_2 | S_1 = e_l, X_1 = x) = \sum_{j_1, \dots, j_{t-1}}^K a_{lj_1} \dots a_{j_{t-1}k} h_{j_1, \dots, j_{t-1}}(x)$$

which is strictly positive for some t because of the irreducibility of $\{Q_t\}$ and the fact that $h_{j_1, \dots, j_{t-1}}(x) > 0$.

2. Analogously can be proven the aperiodicity of $\{Z_t\}$.
3. Irreducibility and aperiodicity ensure that every petite set is small. It can be shown that in the model we consider, each compact set is small. It suffices to prove that for every compact set B such that $\varphi(B) > 0$ and for every bounded Borel set $A = A_1 \times A_2$ with $\varphi(A) > 0$ there exists t such that

$$\inf_{x \in B} P((S_{t+1}, X_{t+1}) \in \{e_k\} \times A_2 | S_1 = e_l, X_1 = x) > 0.$$

Proceeding as in the proof of irreducibility, we need to show that

$$\inf_{x \in B} \sum_{j_1, \dots, j_{t-1}}^K a_{lj_1} \dots a_{j_{t-1}k} h_{j_1, \dots, j_{t-1}}(x) > 0$$

for some t . Continuity of $h_{j_1, \dots, j_{t-1}}(x)$ and irreducibility of $\{Q_t\}$ ensure us that the last infimum is positive (see [2, 3]).

4. We will apply drift criterion of Theorem 15.0.1 (iii). We need to show that there exist $L > 0$, $\beta > 0$ and a function $V(Z) > 1$ such that for $\|Z_{t-1}\| > L$ we have

$$\frac{E(V(Z_t) | Z_{t-1} = (e_l, x)) - V(e_l, x)}{V(e_l, x)} \leq -\beta.$$

Let $V(Z_t) = 1 + X_t^2$. We obtain

$$\begin{aligned} & \frac{E(V(Z_t) | Z_{t-1} = (e_l, x)) - V(e_l, x)}{V(e_l, x)} = \\ & \frac{\sum_{k=1}^K (m_k^2(x) + \sigma_k^2(x) + \kappa A_k^2(x) + 2m_k(x)A_k(x)) E(S_{tk} | S_{t-1} = e_l) - x^2}{1 + x^2} \leq \\ & \frac{\sum_{k=1}^K (m_k^2(x) + \sigma_k^2(x) + \kappa A_k^2(x) + 2m_k(x)A_k(x)) a_{lk}}{x^2} - 1 \end{aligned}$$

The conclusion is obtained by Assumption 8. \square

3. Summary and remarks

We have derived sufficient conditions for geometric ergodicity of a general class of asymmetric nonparametric stochastic processes with stochastic volatility. It is natural to ask, whether other kinds of ergodicity (eg. polynomial ergodicity) can also be related to the skewed SV-CHARME model. It is the purpose of our future work.

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References

- [1] Asai, M. and M. McAleer (2006), Dynamic Asymmetric Leverage in Stochastic Volatility Models, *Econometric Reviews*, 25 453-473.
- [2] Bhattacharia, R. and Lee, C. (1995) On geometric ergodicity of nonlinear autoregressive models. *Statistics & Probability Letters* 22, 311-315.
- [3] Bhattacharia, R. and Lee, C. (1999) Erratum: On geometric ergodicity of nonlinear autoregressive models. *Statistics & Probability Letters* 41, 439-440.
- [4] Cappuccio, N., Lubian, D., Raggi, D., (2004). MCMC Bayesian estimation of a skew-GED stochastic volatility model. *Studies in Nonlinear Dynamics & Econometrics* 8, article 6.
- [5] Danielsson, J. (1994), Stochastic Volatility in Asset Prices: Estimation with Simulated Maximum Likelihood, *Journal of Econometrics*, 64,375-400.
- [6] Danielsson, J. (1998), Multivariate Stochastic Volatility Models: Estimation and a Comparison with VGARCH Models, *Journal of Empirical Finance*,5, 155173.
- [7] Geweke, J., (1994). Comments on bayesian analysis of stochastic volatility. *Journal of Business and Economics Statistics* 12, 371417.

- [8] Glosten, L.R., Jagannathan, R., Runkle, D.E., (1993). On the relation between the expected value and the volatility nominal excess return on stocks. *Journal of Finance*. 46, 1779-1801.
- [9] Hansen, B.E., (1994). Autoregressive conditional density estimation. *International Economic Review* 35 (3), 705-730.
- [10] Harvey, A.C., Ruiz, E., (1993). Multivariate stochastic volatility models. *Review of Economics Studies* 61, 247-264.
- [11] Harvey, A.C., E. Ruiz and N. Shephard (1994), Multivariate Stochastic Variance Models, *Review of Economic Studies*, 61, 247-264.
- [12] Harvey, A., Shephard, N., (1996). The estimation of an asymmetric stochastic volatility model for asset returns. *Journal of Business and Economic Statistics* 14, 429-434.
- [13] Hull, J., White, A., (1987). The pricing of options on assets with stochastic volatility. *Journal of Finance* 3, 281-300.
- [14] Jacquier, E., N.G. Polson and P.E. Rossi (2004), Bayesian Analysis of Stochastic Volatility Models with Fat-tails and Correlated Errors, *Journal of Econometrics*, 122, 185-212.
- [15] Lee, O. and Shin, D.W. (2004) On geometric ergodicity of an AR-ARCH type process with Markov Switching. *Journal of Korean Mathematical Society* 41, 309-318.
- [16] McAleer, M. (2005), Automated Inference and Learning in Modeling Financial Volatility, *Econometric Theory*, 21, 232-261.
- [17] Meyn, S.P. and Tweedie, R.L. (1993) *Markov Chain and Stochastic Stability*. Berlin: Springer Verlag.
- [18] Nelson, D.B. (1991), Conditional Heteroskedasticity in Asset Returns: A New Approach, *Econometrica*, 59, 347-370.
- [19] Shephard, N., Pitt, M.K., (1997). Likelihood analysis of non-Gaussian measurement time series. *Biometrika* 84, 653-667.

- [20] Shephard, N. (1996), Statistical Aspects of ARCH and Stochastic Volatility, in D.R.Cox, D.V. Hinkley and O.E. Barndorff-Nielsen (eds.), Time Series Models in Econometrics, Finance and Other Fields, Chapman & Hall, London, pp. 1-67.
- [21] So, M.K.P., W.K. Li and K. Lam (2002), A Threshold Stochastic Volatility Model, Journal of Forecasting, 21, 473-500.
- [22] Stockis, J.P., Franke, J. and Kamgaing J.T. (2010) On geometric ergodicity of CHARME models. Journal of Time Series Analysis 31, 141-152.
- [23] Taylor, S.J. (1986), Modelling Financial Time Series, Wiley, Chichester.
- [24] Tsiotas, G. (2012) On generalised asymmetric stochastic volatility models. Computational Statistics and Data Analysis 56, 151-172.
- [25] Wang, J.J.J., Chan, J.S.K., Choy, S.T.B., 2011. Stochastic volatility models with leverage and heavy-tailed distributions: a Bayesian approach using scale mixtures. Computational Statistics and Data Analysis 55, 852-862.
- [26] Wiggins, J.B. (1987), Option Values Under Stochastic Volatility: Theory and Empirical Estimates, Journal of Financial Economics, 19, 351-372.